Borderline variants of the Muckenhoupt-Wheeden inequality

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The Hardy-Littlewood maximal operator

Definition

We consider the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy.$$


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Now we add a weight ($dx \rightsquigarrow w(x)dx$), and

$$M : L^1(w) \longrightarrow L^{1,\infty}(w) \quad \Longleftrightarrow \quad w \in A_1.$$

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$$\text{For every weight } w, \quad M : L^1(\mathbf{??}) \longrightarrow L^{1,\infty}(w)$$

?? must be "larger" than w!

Fefferman-Stein inequality

In 1971, C. Fefferman and E. M. Stein, proved the following inequality:

Theorem

For every weight w, it holds that

$$M : L^1(Mw) \longrightarrow L^{1,\infty}(w).$$

Or equivalently,

$$\lambda w(Mf > \lambda) \leq C \int_{\mathbb{R}^n} |f(x)| Mw(x) dx, \quad \forall \lambda > 0.$$

Muckenhoupt-Wheeden conjecture

The conjecture of B. Muckenhoupt and R. Wheeden was that "the same held for every Calderón-Zygmund operator T":

$$M : L^1(Mw) \longrightarrow L^{1,\infty}(w). \quad \text{(FS inequality)}$$

$$T : L^1(Mw) \longrightarrow L^{1,\infty}(w). \quad \text{(MW conjecture)}$$

Muckenhoupt-Wheeden conjecture

This was shown to be FALSE by M. C. Reguera and C. Thiele in 2012

$$H : L^1(Mw) \not\longrightarrow L^{1,\infty}(w).$$

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Question

How can we modify the maximal operator $M \rightsquigarrow M_\varphi$ in the weight so that

$$T : L^1(M_\varphi w) \longrightarrow L^{1,\infty}(w),$$

for every w and Calderón-Zygmund operator T?

We need to make M "larger"...

Orlicz versions of the maximal operator

The Hardy-Littlewood maximal operator can be written as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)| dy = \sup_{x\in Q} \int_{\mathbb{R}^n} |f(y)| \frac{\chi_Q(y) dy}{|Q|}.$$

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The Hardy-Littlewood maximal operator can be written as

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We define

$$M_\varphi f(x) = \sup_{x\in Q} \|f\|_{\varphi(L)(\chi_Q/|Q|)},$$

where φ is a Young function such as:

- $\varphi(t) = t \quad \rightsquigarrow \quad L^1$ norm and $M_\varphi = M$,
- $\varphi(t) = t\log t \quad \rightsquigarrow \quad L\log L$ norm,
- ...

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Larger $\varphi \Rightarrow$ Larger operator $M_\varphi \Rightarrow$ Smaller space $L^1(M_\varphi w) \Rightarrow$

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- ...

Larger φ $\Rightarrow$ Larger operator M_φ $\Rightarrow$ Smaller space $L^1(M_\varphi w)$ $\Rightarrow$ More likely $T : L^1(M_\varphi w) \to L^{1,\infty}(w)$.

Current state of the problem

Question

What is the "least" Young function φ *such that*

$$T : L^1(M_\varphi w) \longrightarrow L^{1,\infty}(w),$$

for every w *and Calderón-Zygmund operator* T*?*

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$$\varphi(t) = t \quad \text{FALSE}$$

(Reguera 2011 / Reguera, Thiele 2012)

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$$\varphi(t) = t(\log t)^\epsilon \quad \text{TRUE}, \ \forall \varepsilon > 0$$

(Pérez 1994 / Hytönen, Pérez 2015, with $C = \frac{1}{\epsilon}$)

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$$\varphi(t) = t \log\log t (\log\log\log t)^\alpha \quad \text{TRUE, } \forall \alpha > 1$$

(D-S, Lacey, Rey 2015, and $C = \frac{1}{\alpha - 1}$)

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$$\varphi(t) = o(t \log \log t) \quad \text{FALSE}$$

(Calderelli, Lerner, Ombrossi 2015)

Current state of the problem

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NEGATIVE RESULTS: With the Hilbert Transform.

POSITIVE RESULTS: With the reduction to sparse operators.

Theorem (D-S, Lacey, Rey 2015)

Suppose the Young function φ satisfies

$$c_\varphi = \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^{2^k})} < \infty.$$

Then, for all C-Z operator T, and any weight w, it holds that $T : L^1(M_\varphi w) \longrightarrow L^{1,\infty}(w)$ with constant c_φ. That is,

$$\sup_{\lambda>0} \lambda w\{Tf > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)|\, M_\varphi w(x)\, dx.$$

Theorem (D-S, Lacey, Rey 2015)

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Then, for all C-Z operator T*, and any weight* w*, it holds that* $T : L^1(M_\varphi w) \longrightarrow L^{1,\infty}(w)$ *with constant* c_φ*. That is,*

$$\sup_{\lambda>0} \lambda w\{Tf > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)| \, M_\varphi w(x) \, dx.$$

The function ψ is called the complementary function of φ, and whenever

$$\varphi(t) = tL(t),$$

with L a logarithmic part ($\log t$, $\log\log t$, $\log\log t(\log\log\log t)^\alpha$...), then essentially

$$\psi^{-1}(t) \approx L(t).$$

Hence, for instance, when

$$\varphi(t) = tL(t) = t\log\log t(\log\log\log t)^{\alpha},$$

we have

$$c_\varphi = \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^{2^k})} \approx \sum_{k=1}^{\infty} \frac{1}{\log\log(2^{2^k})(\log\log\log(2^{2^k}))^{\alpha}} \approx \sum_{k=1}^{\infty} \frac{1}{k(\log k)^{\alpha}} \lesssim \frac{1}{\alpha-1}.$$

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$$\approx \sum_{k=1}^{\infty} \frac{1}{k(\log k)^{\alpha}} \lesssim \frac{1}{\alpha - 1}.$$

Therefore, the theorem states that

$$\lambda w\{Tf > \lambda\} \lesssim \frac{1}{\alpha - 1} \int_{\mathbb{R}^n} |f(x)| \, M_\varphi w(x) \, dx.$$

We also recover the sharp constant of Hytönen-Pérez's result, with

$$\varphi(t) = t(\log t)^{\epsilon}$$

we have

$$c_{\varphi} \approx \sum_{k=1}^{\infty} \frac{1}{(\log 2^{2^k})^{\epsilon}} \approx \sum_{k=1}^{\infty} \frac{1}{2^{k\epsilon}} \approx \frac{1}{\epsilon},$$

and hence

$$\lambda w\{Tf > \lambda\} \lesssim \frac{1}{\epsilon} \int_{\mathbb{R}^n} |f(x)|\, M_{\varphi} w(x)\, dx.$$

How to prove it

It is enough to show that, for every *sparse operator* S,

$$\lambda w\{Sf > \lambda\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)| \, M_\varphi w(x) \, dx,$$

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where

$$Sf(x) = \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x),$$

and $\mathcal{S}$ is a family of dyadic cubes such that, for every $Q \in \mathcal{S}$,

$$\Big| \bigcup_{Q'\in\mathcal{S}\,:\,Q'\subsetneq Q} Q' \Big| \leq \frac{|Q|}{8}.$$

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In fact, by linearity, we reduce to (GOAL)

$$\boxed{w\{1 < Sf \le 2\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)|\, M_\varphi w(x)\, dx.}$$

How to prove it

- We split $\mathcal{S}$ into

$$\mathcal{S}_k = \left\{ Q \in \mathcal{S} : 2^{-k-1} < \frac{1}{|Q|} \int_Q |f| \leq 2^{-k} \right\}.$$

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- By Fefferman-Stein, we can assume that $\mathcal{S}_k = \emptyset$ for $k < 2$, and, for every $k \geq 2$, there is a finite number of layers $\mathcal{S}_k = \mathcal{S}_{k,0} \cup \cdots \cup \mathcal{S}_{k,2^k}$:

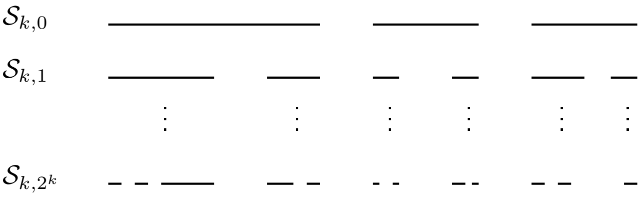
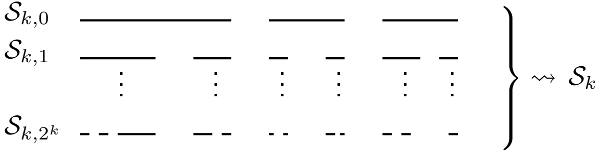
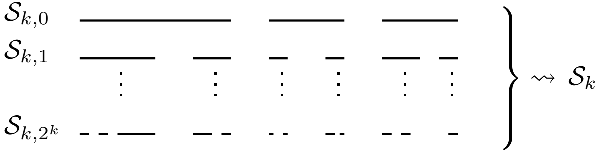


Figure : Layer decomposition of $\mathcal{S}_k$.



This gives a simpler description of the operator:

$$Sf(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x)$$



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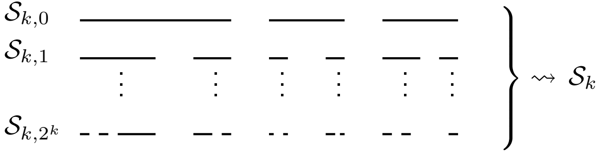
&= \sum_{k=2}^{\infty}\sum_{\nu=0}^{2^k}\sum_{Q\in\mathcal{S}_{k,\nu}} \left(\frac{1}{|Q|}\int_Q |f|\right)\chi_Q(x)
\end{aligned}
$$



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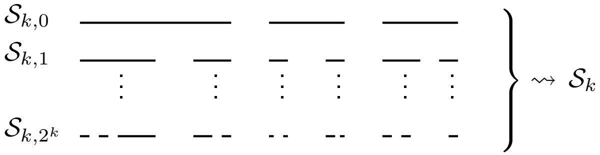

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&= \sum_{k=2}^{\infty} 2^{-k}\sum_{\nu=0}^{2^k}\sum_{Q\in\mathcal{S}_{k,\nu}} \chi_Q(x) \rightsquigarrow \text{overlapping}.
\end{aligned}
$$

The main lemma is the following:

If $S = \sum_{k\geq 2} S_k$, with

$$S_k f(x) = \sum_{Q\in\mathcal{S}_k} \left(\frac{1}{|Q|}\int_Q |f|\right)\chi_Q(x).$$

The main lemma is the following:

If $S = \sum_{k\geq 2} S_k$, with

$$S_k f(x) = \sum_{Q\in\mathcal{S}_k} \left(\frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x).$$

Lemma

For each $k \geq 2$, *if we denote* $\mathcal{E} = \{1 < Sf \leq 2\}$,

$$\int_{\mathcal{E}} S_k f(x) w(x) dx \leq 2^{-k} w(\mathcal{E}) + \frac{C}{\psi^{-1}(2^{2^k})} \int_{\mathbb{R}^n} |f(x)| M_\varphi w(x) dx.$$

Recall our goal was

$$w\{1 < Sf \leq 2\} \lesssim c_\varphi \int_{\mathbb{R}^n} |f(x)|\, M_\varphi w(x)\, dx.$$

Thank you for your attention!

Muchas Gracias!